The Groups of Birational Transformations of Algebraic Curves of Genus 5.

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1. The algebraic curves of genus 5 have been treated in brief by Wiman.* The purpose of the present paper is to show the method by which the normal curves of hyperspace of this genus and their canonical forms in the plane can be projected into each other, and also to find the groups of birational transformations under which they are invariant.

It was shown by Clebsch† that the non-hyperelliptic curve of genus p can always be birationally reduced to a curve of order $p+1$. The general curve of genus 5 is therefore a sextic with five double points, and we distinguish the following cases:

(a) General case. Sextics with five double points.

(b) Curves having a g_3^1, the canonical form being the nodal quintic.

(c) Curves having a g_2^1, the hyperelliptic forms which will not be considered in this paper.

2. (a) The adjoint curves as we shall use them are curves of order $n-3$ which pass once through each double point of C_n. They will have $n(n-3)-2\delta = 2p-2$ variable intersections with C_n and $1/2\,n(n-3)-\delta = p-1$ degrees of freedom. If $\phi_i\,(i=1, 2, \ldots, p)$ are p linearly independent adjoint curves of C_n, the complete system may be written in the form $\sum_{i=1}^{i=p} a_i\phi_i = 0$. This net defines a g_{2p-2}^{p-1} on C_n. Weber‡ has proved that among the ϕ-adjoints of any algebraic curve of genus p there are $1/2(p-2)(p-3)$ quadratic identities. Then, if we think of the ϕ_i as homogeneous point coordinates in linear space

* *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895). This article will be referred to later by W.

† *Vorlesungen über Geometrie*, Vol. I, p. 690 ff.

‡ "Ueber gewisse in der Theorie der Abel'schen Functionen auftretende Ausnahmefälle," *Math. Ann.*, Vol. XIII (1878), pp. 35–48.

of $p-1$ dimensions R_{p-1}, the whole plane may be depicted in R_{p-1} and the C_n is transformed into a hyperspace curve Γ which will be defined in R_{p-1} by the varieties whose equations are the above quadratic identities. This is called the normal curve. The ϕ_i represent spaces of $p-2$ dimensions and an arbitrary ϕ will cut Γ in as many points as the adjoint curve has variable intersections with C_n. Hence, Γ is by definition of order $2p-2$.*

Notation. $G_Q \equiv$ a group of Q points on a curve; also, in another sense, a group of transformations whose order is Q. This usage is quite well established and no ambiguity will arise.

$g_Q^q \equiv$ a set of G_Q's having q degrees of freedom.

$R_n \equiv$ space of n dimensions.

$C_{i,k}^{(p)} \equiv$ a curve of order i and genus p in R_k.

x_i and ϕ_i $(i = 1, 2, \ldots, 5)$ denote point coordinates in R_4.

$P_k \equiv$ a k-fold point.

3. In the general case for $p = 5$, the number of quadratic identities is three and the Γ_{2p-2} is a $C_{8,4}^5$. Now let

$$F_1(\phi_1 \phi_2 \ldots \phi_5) = 0, \quad F_2 = 0, \quad F_3 = 0 \tag{1}$$

be the three identities. Then the net

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0 \tag{2}$$

defines the system of quadrics passing through $C_{8,4}^5$. The five partial derivatives of F may be written

$$\left.\begin{array}{c} \lambda_1 F_{11} + \lambda_2 F_{21} + \lambda_3 F_{31} = 0, \\ \lambda_2 F_{12} + \lambda_2 F_{22} + \lambda_3 F_{32} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \lambda_1 F_{15} + \lambda_2 F_{25} + \lambda_3 F_{35} = 0, \end{array}\right\} \tag{3}$$

where $F_{ik} = \dfrac{\partial F_i}{\partial \phi_k}$, $\quad i = 1, 2, 3; \quad k = 1, 2, \ldots, 5$.

From (3), we obtain two important curves. Eliminating the ϕ_k, we get the discriminant of F which is a ternary quintic in λ. Call it $\Delta_5(\lambda)$. This is the condition for a double point of F and, for all values of λ_i satisfying $\Delta_5 = 0$,

* A more complete discussion of this transformation into R_{p-1} is given by Kraus: "Note über aussergewöhnliche Specialgruppen auf Algebraischen Curven," *Math. Ann.*, Vol. XVI (1880), p. 249. This article will be referred to later by K.

F can be written in terms of four variables instead of five. Eliminating λ, we get the three-by-five matrix in linear functions of the ϕ_k,

$$\left\| \begin{array}{ccccc} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} \end{array} \right\| = 0. \tag{4}$$

The values of ϕ_k satisfying (4) are the coordinates of the vertices of the R_4 cones that exist when the λ_i satisfy $\Delta_5 = 0$. After the transformation into terms of four variables, the vertex is

$$\phi_1' = \phi_2' = \phi_3' = \phi_4' = 0.$$

A curve D_{10} is the locus of these vertices and is the condition that (3) shall be consistent in λ, while Δ_5 is the condition that (3) shall be consistent in ϕ. Thus D_{10} and Δ_5 are in (1, 1) correspondence. Δ_5 is of further importance from the fact that unless the transformations of F leave F_4 absolutely invariant our C_6 which is the plane projection of $C_{8,4}$ can have no transformations except those of this quintic, whose equation is expressible as a symmetric determinant. It must be kept in mind, therefore, that C_6, $C_{8,4}$, Δ_5 and D_{10} will be invariant under groups of transformations of the same order. The complete set of equations obtained by eliminating λ from (4) may be written in the matrix

$$\left.\left\| \begin{array}{ccccc} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} \end{array} \right\| = 0 \ \begin{array}{c}\text{(all elements being linear}\\ \text{in five variables).}\end{array} \right\} \tag{5}$$

Of the ten equations shown here, only three are independent. For, if we have

$$\left.\begin{array}{l} (F_{11}F_{22} - F_{21}F_{12})F_{33} + (F_{31}F_{12} - F_{11}F_{32})F_{23} + (F_{21}F_{32} - F_{31}F_{22})F_{13} = 0, \\ (F_{11}F_{22} - F_{21}F_{12})F_{34} + (F_{31}F_{12} - F_{11}F_{32})F_{24} + (F_{21}F_{32} - F_{31}F_{22})F_{14} = 0, \\ (F_{11}F_{22} - F_{21}F_{12})F_{35} + (F_{31}F_{12} - F_{11}F_{32})F_{25} + (F_{21}F_{32} - F_{31}F_{22})F_{15} = 0, \end{array}\right\} \tag{6}$$

the remaining seven equations follow by the proportionality of columns 3, 4 and 5, which is evident from these three. But the system of equations is restricted and a C_{27} is not defined by them, as would be the case if they were arbitrary. For example, columns 3, 4, 5 need not be proportional if 1 and 2 are, for then the bracketed expressions in (5) vanish.

Let the order be $27 - m$. Let V_{ik} denote an i-dimensional variety of order k in R_4. Then each equation in (4) defines a V_{33}. Columns (1, 2) define a V_{23}, and for all points of this V_{23}, (6) may be satisfied without all of (5) being

satisfied. (3, 4, 5) would be an exception. Hence, the order of the curve defined by these three columns and V_{23} is the value of m. Now (3, 4), (3, 5), (4, 5) define V'_{23}, V''_{23}, V'''_{23} respectively, but, excluding the vanishing of column (3), (4, 5) is a consequence of (3, 4) and (3, 5). (V'_{23}, V''_{23}) and (V'_{23}, V'''_{23}) define C'_9 and C''_9 respectively. Then, excluding (3) as mentioned above, we have

$$m = 9 + 9 - 1 = 17,$$

and the order of the curve represented by (5) is 10. The curves $D_{10,4}$ and $C_{8,4}$ can not intersect. If they do intersect, the three F_i pass through a vertex of an R_4 cone, and $C_{8,4}$ must then have a double point. $C_{8,4}$ has no double points, because its projection in R_3 would be $C^{(5)}_{6,3}$, but no such curve exists.

4. If $C_{8,4}$ be projected from a point upon it into R_3, the resulting curve will be of order 7. Similarly, if $C_{7,3}$ be projected from a point upon it into R_2, the result is a sextic. Since the genus is preserved by each projection, $C_{6,2}$ must have five double points. Both projections can be made at once from two points ξ and η on the R_4 curve. A plane π through ξ and η can, in general, cut $C_{8,4}$ in only one other point P. The image of P in the plane of projection π' will be the point common to π and π'; the double points of $C_{6,2}$ will appear when and only when π cuts $C_{8,4}$ in four points. There will evidently be five such positions of π. Let ρ, λ, μ, σ be parameters and π' be defined by $y_4 = y_5 = 0$. Then the equations of transformation are

$$\rho x_i = \lambda \xi_i + \mu \eta_i + \sigma y_i \qquad (i = 1, 2, \ldots, 5), \tag{7}$$

where y_i are the plane coordinates. Since y_4 and y_5 are zero in π', we find

$$\lambda/\rho = (\eta_5 x_4 - \eta_4 x_5)/(\xi_4 \eta_5 - \xi_5 \eta_4), \qquad \mu/\rho = (\xi_4 x_5 - \xi_5 x_4)/(\xi_4 \eta_5 - \xi_5 \eta_4).$$

Then

$$\sigma y_i = (\eta_4 x_5 - \eta_5 x_4)\xi_i + (\xi_5 x_4 - \xi_4 x_5)\eta_i + (\xi_4 \eta_5 - \xi_5 \eta_4) x_i. \tag{8}$$

Making substitution (7) in $F_1 = 0$, we have

$$\lambda^2 F_1(\xi) + \mu^2 F_1(\eta) + \sigma^2 F_1(y) + 2\lambda\mu P_1(\xi, \eta) + 2\lambda\sigma P_1(\xi, y) + 2\mu\sigma P_1(\eta, y) = 0. \tag{9}$$

Since ξ and η are on $C_{8,4}$, $F_1(\xi) = 0$ and $F_1(\eta) = 0$.

Then we may write the equation in the form

$$\sigma^2 F_1(y) + 2\lambda\mu P_1 + 2\lambda\sigma P'_1 + 2\mu\sigma P''_2 = 0. \tag{10}$$

Similarly, for F_2 and F_3 we have

$$\sigma^2 F_2(y) + 2\lambda\mu P_2 + 2\lambda\sigma P_2' + 2\mu\sigma P_2'' = 0, \tag{11}$$

$$\sigma^2 F_3(y) + 2\lambda\mu P_3 + 2\lambda\sigma P_3' + 2\mu\sigma P_3'' = 0. \tag{12}$$

From the third of these equations,

$$\lambda = -\frac{\sigma^2 F_3(y) + 2\mu\sigma P_3''}{2\mu P_3 + 2\sigma P_3'}. \tag{13}$$

Make this substitution in the first two and put $\mu = 1$.

$$\begin{aligned}\sigma^3 [F_1(y) P_3' - F_3(y) P_1'] + \sigma^2 [P_3 F_1(y) - P_1' F_3''(y) - P_1' P_3'' + P_3' P_1''] \\ + \sigma[(P_1'' P_3 + P_1 P_3'')] = 0.\end{aligned} \tag{14}$$

$$\begin{aligned}\sigma^3 [F_2(y) P_3' - F_3(y) P_2'] + \sigma^2 [P_3 F_2(y) - P_2 F_3(y) - P_2' P_3'' + P_3' P_2''] \\ + \sigma[(P_2'' P_3 + P_2 P_3'')] = 0.\end{aligned} \tag{15}$$

Eliminate σ and we have

$$\begin{vmatrix} F_1(y)P_3' - F_3(y)P_1' & \begin{matrix} P_3F_1(y) - P_1F_3(y) \\ - P_1'P_3'' + P_3'P_1'' \end{matrix} & P_1''P_3 + P_1P_3'' & \\ & F_1(y)P_3' - F_3(y)P_1' & \begin{matrix} P_3F_1(y) - P_1F_3(y) \\ - P_1'P_3'' + P_3'P_1'' \end{matrix} & P_1''P_3 + P_1P_3'' \\ F_2(y)P_3' - F_3(y)P_2' & \begin{matrix} P_3F_2(y) - P_2F_3(y) \\ - P_2'P_3'' + P_3'P_2'' \end{matrix} & P_2''P_3 + P_2P_3'' & \\ & F_2(y)P_3' - F_3(y)P_2' & \begin{matrix} P_3F_2(y) - P_2F_3(y) \\ - P_2'P_3'' + P_3'P_2'' \end{matrix} & P_2''P_3 + P_2P_3'' \end{vmatrix} = 0. \tag{16}$$

This is a plane C_8 in (y_1, y_2, y_3). It must, however, have a quadratic factor, because $C_{8,4}$ goes into a $C_{6,2}$ when the centers of projection are on the curve. The other factor is the $C_{6,2}$.

To project $C_{6,2}$ into $C_{8,4}$, we use the same equations of transformation as before, but the parameters must be expressed in terms of (y, ξ, η) instead of (x, ξ, η). In order to do this, solve (4), (5), (6) for $\lambda : \mu : \sigma$. This is possible by virtue of (10). We may eliminate the terms in $\lambda\mu$ and $\mu\sigma$ by multiplying

$$(4) \text{ by } \begin{vmatrix} P_2 & P_2'' \\ P_3 & P_3'' \end{vmatrix}, \quad (5) \text{ by } -\begin{vmatrix} P_1 & P_1'' \\ P_3 & P_3'' \end{vmatrix}, \quad (6) \text{ by } \begin{vmatrix} P_1 & P_1'' \\ P_2 & P_2'' \end{vmatrix}$$

and adding. Then

$$\frac{\lambda}{\sigma} = 1/2\, \frac{(P_3 P_2'' - P_2'' P_3) F_1 + (P_1 P_3'' - P_3 P_1'') F_2 + (P_2 P_1'' - P_1 P_2'') F_3}{P_1'(P_2 P_3'' - P_3 P_2'') + P_2'(P_3 P_1'' - P_1 P_3'') + P_3'(P_1 P_2'' - P_2 P_1'')}. \tag{17}$$

By a similar procedure we obtain

$$\frac{\mu}{\sigma} = 1/2 \frac{(P_2 P_3' - P_3 P_2') F_1 + (P_3 P_1' - P_1 P_3') F_2 + (P_1 P_2' - P_2 P_1') F_3}{P_1''(P_3 P_2' - P_2 P_3') + P_2''(P_1 P_3' - P_3 P_1') + P_3''(P_2 P_1' - P_1 P_2')}. \quad (18)$$

The F's are quadratic in y and P_i', P_i'' are linear. Hence, letting $\rho = 1$, the equations of transformation are of the form

$$x_i = C_3(y)\xi_i + C_3'(y)\eta_i + C_2(y) y_i, \qquad i = 1, 2, \ldots, 5. \quad (19)$$

The form of the general transformation in the plane may be found by first projecting C_6 into R_4. By (19) this is a cubic relation. Next make a transformation in R_4 that leaves $C_{8,4}$ invariant. This transformation will always be linear. Finally, project back into R_2. By (8) this also is a linear relation. Hence, the successive projections are expressed by the equations

$$y' = f_1(x') = \phi_1(x) = \psi_3(y). \quad (20)$$

This shows that there are no transformations of higher degree than the cubic.

5. Δ_5 will have double points for all values of λ_i that make its first minors vanish. But all these minors can be made to vanish by making four properly chosen ones vanish. The net of R_4 quadrics degenerates into a net of R_4 cones K' when $\Delta_5 = 0$. The elements of any K' are planes concurrent at the vertex $P_v \equiv x_1' = x_2' = x_3' = x_4' = 0$. By means of the four conditions on the minors mentioned above, we may fix four points on K_2', viz., P_1, P_2, P_3, P_4. Then, $P_v P_1 P_2$, $P_v P_2 P_3$, $P_v P_3 P_4$ can be made to define three elements (planes) in the same R_3. Of the general K_2 only two planes can lie in the same R_3, and if three elements are so placed, the K_2' degenerates and all of the elements will lie in two R_3's which intersect in a plane. In this plane is a line l through which all elements of the degenerate K_2' pass. An R_3 section of K_2' is an ordinary quadric surface H_2, and the section of the degenerate K_2' is the quadric cone K_2. The vertices of K_2's thus cut from K_2' will be collinear with P_v. But l contains P_v. Hence, the line of vertices of the K_2's intersects l and both these lines lie in the plane common to the two R_3's mentioned above. l is defined by $x_1'' = x_2'' = x_3'' = 0$. Now for certain values of λ_i the first minors of Δ_5 vanish, showing a double point of Δ_5, and the point P traces Δ_5 continuously while its image P' describes D_{10}. Then, since P' is replaced by a whole line when P reaches a double point, it is evident that this line corresponds to the double point of Δ_5. Whenever a double point of Δ_5 exists, then, for the values of λ_i that are the coordinates of this point, F can be expressed in terms of three variables

(x_1, x_2, x_3). The most general quadratic in three variables can be written in the form
$$2x_1 x_2 - x_3^2 = 0.$$
The linear form $ax_1 + bx_2 + cx_3 = 0$ will cut this conic in (x_1', x_2', x_3') and (x_1'', x_2'', x_3''). Regarding the x's as adjoint curves, the tangent C_3's may be written
$$x_1' x_2 + x_2' x_1 - x_3' x_3 = 0, \qquad x_1'' x_2 + x_2'' x_1 - x_3'' x_3 = 0.$$
Then F and $C_{6,2}^{(5)}$ take the form
$$2(x_1' x_2 + x_2' x_1 - x_3' x_3)(x_1'' x_2 + x_2'' x_1 - x_3'' x_3) - (ax_1 + bx_2 + cx_3)^2 = 0.$$
An equation like this is possible for each double point of Δ_5. The two Abel forms or contact curves appear in the product term of the equation. Since there is an infinite number of curves of the form $ax_1 + bx_2 + cx_3 = 0$, there is also an infinite number of contact curves for each double point of Δ_5. When Δ_5 consists of five straight lines, the number of double points is ten. Hence, there may be ten systems of contact curves. By Kraus' (K) proof there are but three linearly independent systems. If three such systems be known, the remaining ones can be obtained linearly in terms of them. For example, if we have
$$F_1 \equiv a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0,$$
$$F_2 \equiv b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 = 0,$$
$$F_3 \equiv c_3 x_3^2 + c_4 x_4^2 + c_5 x_5^2 = 0,$$
Δ_5 will consist of five straight lines. There should be therefore ten systems of contact curves. Any two of the five variables may be eliminated by linear combinations of the above F_i. Thus we get the ten systems and there are no more. By the same procedure with any three linearly independent F_i in three variables, we can get as many systems as there are double points of Δ_5. Wiman (W) states that Kraus' proof is incorrect, but he overlooks the fact that Kraus says "linearly independent" systems. Since a double point of Δ_5 calls for a linear factor of D_{10}, the D_{10} in the above illustration must consist of ten straight lines. They are of the form $x_1 = x_2 = x_3 = 0$. Obviously four such lines will go through the point $x_1 = x_2 = x_3 = x_4 = 0$. The ten lines will be concurrent by fours in five such points and will constitute the simplex of reference in R_4.

Our C_6 has a g_4^1 for each of the five double points and also for each set of four double points. Through each P_2 there is a pencil of straight lines cutting out a g_4^1 and through each set of four points can be passed a C_2 cutting C_6 in a g_4^1.

Now, by the Brill-Noether reciprocity theorem, we have the equations

$$Q+R=2p-2, \qquad Q-R=2(q-r).$$

Since in this case $p=5$ and $R=4$, we find

$$Q=R=4 \quad \text{and} \quad q=r=1.$$

Now take any G_4 of the above g_4^1 as basis points. Pass a C_{n-3}, *i. e.*, a C_3, through them and the five double points. The residual is a g_4^1 from which we may in turn select a G_4 and determine the g_4^1 first used.

These g_4^1's have a simple relation to D_{10}. Project $C_{8,4}$ from a point on D_{10} into R_3. This gives a $C_{8,3}$ lying on a quadric surface. The equation of this quadric is the same as that of $F=0$ in R_4, where it has been expressed in terms of four variables. This $C_{8,3}$ is of type (4, 4) on the quadric. The other possibilities are eliminated because the (7, 1) could be projected into a $C_{8,2}$ with a P_7, which would make its genus 0. The (6, 2) goes into a $C_{8,2}$ with a P_6 and a P_2 which is of genus 5, but on account of the P_6 a g_2^1 exists, making the curve hyperelliptic. The (5, 3) has a g_3^1 by the pencil of lines through P_5 of the $C_{8,2}$. This case has been disposed of in the nodal C_5. By examining the (4, 4), we find that it must have four double points in order to make the genus 5, for it projects into a $C_{8,2}$ with two P_4's, which with no further multiple points would make the genus 9. Project the curve $C_{8,3}$ from one of these four points P_2' into R_2 and we get a C_6 with five P_2, for the other three P_2's give P_2's in the plane and each of the two generators through P_2' cuts the $C_{8,3}$ in two points distinct from P_2. This provides two more double points, which with the three mentioned above make the genus 5. These four double points in R_3 show that from every point of $D_{10,4}$ four bisecants can be drawn to $C_{8,4}$. No two of the above double points can lie on the same generator of the R_3 quadric, for in that case the proper number of double points would not appear in C_6 if one of these two points should be used as the center of projection.

These g_4^1's are grouped in pairs, for, consider P_2' again as center of projection. Call the two generators through it A and B. All the generators of the A system cut B and project into a pencil of lines through the P_2 determined by B. Similarly for the B generators.

If F be expressed in terms of three variables, the R_3 quadric is a cone of order 2, so that the two systems of generators and therefore the two g_4^1's coincide.

The conic through four P_2's cutting out a g_4^1 may be obtained by passing a plane through two P_2's of the curve on the above quadric. Fix the plane by

a simple point P_1 and project upon the plane of C_6 as before. The plane of section must cut $C_{8,3}$ in eight points. There are two at each P_2 and four simple points including P_1. These four will in general project into four simple points. The two P_2's through which the section was made give two P_2's of $C_{6,2}$. The two generators A and B determine two P_2's as before, and the section of the quadric is thus seen to project into a conic through four P_2's of C_6. The P_2 of $C_{8,3}$ not used above projects into the fifth P_2 of C_6.

6. If the normal curve $C_{8,4}$ be linearly transformed into itself, the system

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0 \tag{21}$$

will also go into itself by the same transformation. For, $C_{8,4}$ is the complete intersection of the F_i and, since those points of the system of varieties that constitute $C_{8,4}$ remain invariant as a whole, the transformed system will pass through $C_{8,4}$. It is therefore identical with F. By this transformation, the F_i may not remain individually fixed, in which case the λ_i are linearly transformed in such a way as to put Δ_5 into itself, because it, the condition for double points of F, must remain unchanged when F remains unchanged. But in case F_i goes into F_i, the λ_i will not be altered; *i. e.*, the individual points of Δ_5 remain fixed. Since Δ_5 and D_{10} are in (1, 1) correspondence, D_{10} will also go into itself point by point with the exception of the lines corresponding to the double points of Δ_5. These lines go individually into themselves, but the points may be permuted by one or more homologies. For, if the points of D_{10} remain fixed, it can not be a proper R_4 curve. If it were, every point of R_4 would be fixed and the transformation would be an identity. D_{10} therefore consists of a point and an R_3 curve or a line and an R_2 curve. If a point and an R_3 remain fixed, call them O and R_3'. These must be the center and invariant R_3 of a homology. Any R_3 through O is invariant as a whole, for it cuts R_3' in a plane. Take R_3' and any four R_3's through O as the simplex of reference and the equation of F_i will be reducible to

$$a_i x_1^2 + f_i (x_2 x_3 x_4 x_5) = 0, \tag{22}$$

where $x_1 = 0$ is the equation of R_3'. The transformation is

$$x_1 = -x_1', \qquad x_i = x_i', \qquad i = 2, 3, 4, 5. \tag{23}$$

In this case $\Delta_5 \equiv C_1 . C_4$. The four double points thus formed require four systems of ϕ-curves. When even one such system exists, the $C_{6,2}^{(5)}$ is reducible to the form having a tacnode and three collinear double points (K). Hence, when Δ_5 is point by point invariant, the C_6 must be thus far restricted in order to be invariant under a linear G_2.

When a line and an R_2 curve remain fixed, the plane will be common to the invariant R_3's of two homologies whose centers lie on the fixed line. The F_i will be of the form

$$a_i x_1^2 + b_i x_2^2 + \psi_i(x_3 x_4 x_5) = 0, \tag{24}$$

if we take three points in the fixed plane and two points on the fixed line as the vertices of the simplex of reference. The fixed plane is defined by $x_1 = 0$, $x_2 = 0$ and the line by $x_3 = x_4 = x_5 = 0$. The centers of the two homologies are

$$x_1 = x_3 = x_4 = x_5 = 0; \quad x_2 = x_3 = x_4 = x_5 = 0.$$

The equations of transformation are

$$x_1 = -x_1', \quad x_2 = -x_2', \quad x_i = x_i', \qquad i = 3, 4, 5. \tag{25}$$

According as Δ_5 is proper or degenerate, we have the following cases:

(a) Δ_5 *is a proper* C_5. The maximum number of double points of Δ_5 and of the corresponding lines of D_{10} is six. The transformations depend upon the coefficients of F_i.

(b) $\Delta_5 \equiv C_1 . C_4$. See equations (22) and (23).

(c) $\Delta_5 \equiv C_2 . C_3$. $F_i \equiv f_i(x_1 x_2) + \psi_i(x_3 x_4 x_5) = 0$. The intersections of C_2 and C_3 call for six lines of D_{10}. There will be seven if C_3' has a double point.

(d) $\Delta_5 \equiv C_1 . C_1' . C_3$. See equations (24) and (25).

(e) $\Delta_5 \equiv C_1 . C_2 . C_2'$. Eight systems of ϕ-curves.

$$F_i \equiv a_i x_1^2 + f_i(x_2 x_3) + \psi_i(x_4 x_5) = 0.$$

(f) $\Delta_5 \equiv C_1 . C_1' . C_1'' . C_2$. Nine systems of ϕ-curves and nine corresponding lines of D_{10}. The tenth line does not correspond to a double point of Δ_5 and therefore is not the vertex of a cone through the normal curve.

(g) $\Delta_5 \equiv 5\, C_1$. D_{10} consists of ten straight lines.

$$F_i \equiv a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 = 0.$$

From any three linearly independent forms of F_i we can eliminate any pair of variables x_r and x_s and thus obtain the ten relations among five variables in threes.

We will illustrate two tacnodal cases, namely (a) and (d).

For (a) write the equation of the general C_3 (the ϕ-curve) through (0, 1, 0), (1, 1, 0), (1, 0, 0) and tangent to $y = 0$ at (0, 0, 1). The quadratic equation in x_i defines a C_6 with a tacnode at (0, 0, 1) and having three double points on $z = 0$. The C_3 is

$$K(x^2 y - xy^2) + Lx^2 z + Py^2 z + Qyz^2 + Rxyz = 0.$$

Then pass to R_4 by the transformation

$$x_1 = xy(x-y), \quad x_3 = y^2 z, \quad x_5 = xyz,$$
$$x_2 = x^2 z, \qquad x_4 = yz^2.$$

$$F_1 \equiv x_2 x_3 - x_5^2 = 0, \quad F_2 \equiv x_1 x_4 + x_3 x_5 - x_2 x_3 = 0.$$

Considering these two identities, the general quadratic in x may be written with thirteen terms:

$$F_3 \equiv A x_1^2 + B x_2^2 + C x_3^2 + D x_4^2 + E x_5^2 + F x_1 x_2 + G x_1 x_3 + H x_1 x_5$$
$$+ K x_2 x_4 + L x_3 x_4 + M x_2 x_5 + P x_3 x_5 + N x_4 x_5 = 0.$$

This will be invariant under $x_1 = x_4'$, $x_4 = x_1'$, if $A = D$, $F = K$, $G = L$ and $H = N$. Since this transformation can be reduced to a single change of sign, F_3 is the equation of the required sextic.

For (d), write the equation of the general C_3 through $(0, 0, 1)$, tangent to $x = 0$ at $(0, 1, 0)$ and to $y = 0$ at $(1, 0, 0)$. It is

$$K x^2 y + M x y^2 + N x z^2 + Q y z^2 + R x y z = 0.$$

Then let

$$x_1 = x^2 y, \quad x_2 = x y^2, \quad x_3 = x z^2, \quad x_4 = y z^2, \quad x_5 = x y z.$$

Two identities will be

$$F_1 \equiv x_1 x_4 - x_5^2 = 0, \quad F_2 \equiv x_2 x_3 - x_4^2 = 0.$$

Similar to the preceding, we have $F_3 = 0$, but to make it invariant under $x_1 = -x_1'$, $x_4 = -x_4'$, there must be no terms linear in x_1 or x_4.

$$F_3 \equiv A x_1^2 + B x_2^2 + C x_3^2 + D x_4^2 + E x_5^2 + L x_2 x_5 + N x_3 x_5 = 0.$$

This is the equation of the sextic. Δ_5 is

$$\begin{vmatrix} 2A\lambda_3 & \lambda_1 \\ \lambda_1 & 2D\lambda_3 \end{vmatrix} \cdot \begin{vmatrix} 2B\lambda_3 & \lambda_2 & L\lambda_3 \\ \lambda_2 & 2C\lambda_3 & N\lambda_3 \\ L\lambda_3 & N\lambda_3 & -2(\lambda_1 + \lambda_2 - E\lambda_3) \end{vmatrix} = 0.$$

7. The normal curve in R_{p-1} is defined by the quadratic identities among p linearly independent adjoint curves of $C_{n,2}^{(p)}$. For $p = 5$, as already shown, this curve is of order 8. If $C_{8,4}$ be projected into R_2 from two points upon it and the image of one of these points used as a basis point in addition to the five double points, we get the system of adjoint curves,

$$\phi \equiv a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3 + a_4 \phi_4 = 0,$$

where the ϕ_i are linearly independent and may be regarded as homogeneous point coordinates in R_3. By means of these ϕ_i as functions of x, y, z the points

of the plane may be uniquely pictured as a surface in R_3 on which lies a C_7 projective with C_6. The curve is of order 7, because the adjoint curves have

$$18 - 2 \cdot 5 - 1 = 7$$

variable intersections with C_6. From these relations among the three curves, we know that $C^{(5)}_{6,2}$ and $C^{(5)}_{7,3}$ may always be considered as projections of the normal $C_{8,4}$.

If $C_{8,4}$ be projected into R_2 from O_1 and O_2, two points on the curve, the images of the two centers of projection will be the two simple intersections of C_6 and the C_2 passing through the five double points. For, the ∞^2 system of R_3's through O_1, O_2 defines a g^2_6 on C_8. Take O_1 as center and project the system into R'_3. The resulting system of planes through O'_2 defines a g^2_6 on $C_{7,3}$. Then projecting from O'_2, we get the ∞^2 system of straight lines in the plane defining a g^2_6 on C_6. Hence, the adjoint curves represented by the R_3's through O_1 and O_2 must consist of an arbitrary straight line and the C_2 defined by the five double points. The two remaining points in which C_2 cuts C_6 are invariant for the above system of R_3's and are therefore the images of the only two fixed points defined by the system, *i. e.*, O_1 and O_2.

If four of the points common to an R_3 and $C_{8,4}$ lie in a plane, the other four lie in a plane. Let G_4 be such a set of four points. The R_3 containing them has one degree of freedom and therefore the other four points constitute a Γ_4, where all the Γ_4's form a γ^1_4. Now, by the Brill-Noether theorem, any Γ_4 of γ^1_4 may be used as basis points of a system of R_3's which, by reason of the one degree of freedom, will define a g^1_4, and to this g^1_4 belongs the above G_4. The Γ_4's have one degree of freedom and lie therefore in a plane. Q. E. D.

There are ∞^2 planes through each point of $C_{8,4}$ cutting the curve in three additional points.

Through any two points O_1, O_2 of C_8 may be passed five planes cutting the curve in two other points. Now let O_1 be held fast while O_2 describes C_8. The set of five planes thus take ∞^1 positions, but still pass through O_1.

Therefore there are ∞^1 triads of points co-planar with each point of $C_{8,4}$. Q. E. D.

There are ∞^1 planes through each point of D_{10} cutting $C_{8,4}$ in four points. To see this, project C_8 from O of D_{10}. We get a $C_{8,3}$ of species (4, 4) on a hyperboloid H_2. Therefore a plane through O and a generator of H_2 cuts C_8 in four points. O and the ∞^1 generators define the ∞^1 planes. Q. E. D.

By projecting C_8 from P_1, a point on the curve, we get a $C_{7,3}$. With this curve is associated a ruled surface of trisecants. For, any three points co-planar with P_1 have for images three points in the line common to this plane and the R_3 of section. Therefore corresponding to each of the ∞^1 planes through P_1 containing three other points of C_8 is a trisecant of $C_{7,3}$. Since D_{10} is the locus of points from which C_8 can be projected into a (4, 4) curve on H_2, all the planes through P_1 defining a G_4 must cut D_{10}.

Now turn the plane on the line OP_1 until it contains the reciprocal G_4. It will then define the trisecant B which will intersect A at O', because P_1O is common to the two planes defining the reciprocal G_4's. O' is therefore a double point on $D_{10,3}$; the same argument holds for every position of O', since O is an arbitrary point on $D_{10,4}$. $D_{10,3}$ is therefore a double curve on S, the ruled surface of trisecants. Through $P_1\,P_2$ will pass five planes that cut $C_{8,4}$ in four points. Hence through P_2' in R_3 will pass five generators of S, which means that $C_{7,3}$ is a five-fold curve on S. In general, there will not be an R_3 tangent to $C_{8,4}$ in four points, but if F is expressed in terms of three variables, *i. e.*, if the curve lies on a $K_{2,4}$ having a line D_1 for vertex, we have ∞^1 such R_3's. The two G_4's defined by a tangent R_3 are coincident and lie in the R_2 of contact. Project $C_{8,4}$ from one of the four points of tangency. The images in R_3' of the other three points are the intersections of $C_{7,3}$ and a trisecant. These points of C_7 have a common tangent plane, namely, the intersection of the tangent R_3 with R_3'. This trisecant corresponds to two coincident G_4's and must therefore count for two. It is a double torsal generator of S (W). Project $C_{7,3}$ from one of the above points of tangency and the resulting curve is a $C_{6,2}$ with a tacnode corresponding to the other two points of tangency, while the two successive centers of projection appear as the residual intersections of C_6 and the tacnodal tangent. Again, if we project $C_{8,4}$ from a point of the line vertex D_1 of $K_{2,4}$, $C_{8,3}$ will lie on a quadric cone cutting each generator four times and not passing through the vertex. Projecting $C_{8,3}$ into R_2 from any point of the cone, we get a plane C_8 having a tacnode with four branches which is equivalent to twelve double points. Therefore, in order to have the proper genus, the $C_{8,3}$ must have four actual double points. By projecting $C_{8,3}$ from one of these double points, we get a plane C_6 with a tacnode. The images of the other three double points are collinear, because by Kraus' proof, if the adjoint curves of a C_{p+1} touch the curve at $p-1$ points, $1/2\,(p-4)\,(p+1)$ of the double points lie on a C_{p-4}. In our case $p=5$. Hence the three collinear double points.

These three being collinear, the four double points of $C_{8,3}$ must lie in a plane.

We have seen that $D_{10,8}$ is a double curve and $C_{7,8}$ is a five-fold curve on S. To find the order of the surface, let g be a trisecant of $C_{7,8}$, *i. e.*, a generator of S. Let P_1, P_2, P_3, D_1 be the points in which g cuts $C_{7,8}$ and $D_{10,8}$ respectively. Pass a plane through g. It will be tangent to S at some point T which is a double point of the plane section C_n, and g is one of the branches through T. C_n evidently consists of a C_{n-1} and the line g cutting C_{n-1} four times at each of the points P_1, P_2, P_3, and once at D_1 and T. Therefore,

$$n - 1 = 4 \,.\, 3 + 1 + 1,$$

whence S is of order 15. The double points of an arbitrary plane section arise from the ten points of the double curve $D_{10,3}$ and the seven points of the five-fold curve $C_{7,3}$ that lie in the plane. This shows

$$10 + 7 \,.\, \frac{5 \,.\, 4}{2} = 80 \text{ double points.}$$

Now every torsal generator reduces the order of $D_{10,3}$ by 1, but adds 2 to the number of double generators, because the torsal generator itself is a double line along which the surface has a line of self-contact; any plane section has a tacnode at its point of intersection with this generator. Let δ be the number of them, then $p = 1/2 \,.\, 14 \,.\, 13 - 80 - \delta = 11 - \delta$.

When one of the F_i can be written in terms of three variables, then

$$D_{10} \equiv D_9 \,.\, D_1,$$

in which D_1 corresponds to a double point of Δ_5. Since both curves are continuous D_1 must cut D_9. The double point so formed does not call for a linear factor of Δ_5. We have already seen that there are ∞^1 R_3's through D_1 tangent to $C_{8,4}$. The planes of tangency are uniquely determined by D_1 and the points of C_8. Hence, any plane through D_1 and a point of the normal curve defines a G_4 on the normal curve.

Forms of S_{15}. [1] When Δ_5 is a proper curve, at most six double points may exist and an equal number of double torsal generators. Thus the maximum value of δ is 6 and the minimum genus of S_{15} is 5. The torsal generators may or may not intersect.

[2] If we have $F_1(x_1, x_2, x_3) = 0$, $F_2(x_1, x_2, x_4) = 0$, $F_3(x_1, \ldots, x_5) = 0$, the normal curve lies on two quadric cones of R_4 whose vertices are $x_1 = x_2 = x_3$ and $x_1 = x_2 = x_4$. These two lines lie in the plane (x_1, x_2) and therefore will

intersect. Now, if we find Δ_5 it is evident from the forms of F_i that λ_3 may be factored out from either the fifth column or the fifth row. Hence, $\Delta_5 \equiv C_4 . C_1$. The intersections of C_4 with C_1 require that D_{10} have four linear factors each of which will be the vertex of a cone passing through $C_{8,4}$. Since a line of Δ_5 corresponds to a double point of D_{10}, the four lines must pass through that double point. It can be shown directly from the equations that the four lines are concurrent. The preceding net may be written

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 [f(x_1, x_2, x_3, x_4) + x_5(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) + x_5^2] = 0,$$

or putting $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 2X$,

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 [f(x_1, x_2, x_3, x_4) - X^2 + (x_5 + X)^2] = 0.$$

Then let $x_i = x_i'$, $i = 1, 2, 3, 4$. $X + x_5 = x_5'$. The net is invariant under $x_i' = x_i''$, $i = 1, 2, 3, 4$. $x_5' = -x_5''$. The center O of this perspective transformation is $x_1'' = x_2'' = x_3'' = x_4'' = 0$; the fixed space is $x_5'' = 0$. It is evident that the lines through O cutting $C_{8,4}$ in one point P_1 must cut it again in P_1'. Since D_{10} is invariant under the transformation, the four lines which are a part of it in this case must also pass through O. Let D_1, D_1', D_1'', D_1''' be the lines of D_{10}. Project $C_{8,4}$ from P_1 and find the surface of trisecants. $P_1 O$ will cut C_8 again in P_2. Join O to another point P_3, and a fourth point P_4 will be determined. There are evidently ∞^1 planes through $P_1 O$ cutting the normal curve in four points. $P_1 P_2$ are included in each G_4 of the g_4^1. Now as P_3 describes C_8 the plane of the G_4 will successively contain D_1, D_1', D_1'', D_1'''; in such cases the G_4 consists of the points of tangency of an R_3. The corresponding trisecant is a double torsal generator. These four double torsal generators intersect in the R_3 image of O, *i. e.*, in O'. The image of P_2 will be at O' in every case and $C_{7,3}$ will cut each trisecant in two points aside from O'. Since there are ∞^1 generators of S_{15} passing through O', the surface must consist in part of a cone K_k with O' as a vertex. $C_{7,3}$ evidently lies on this cone. A plane through two generators of K_k contains the five points on these generators; and since a plane through the vertex of a cone can cut the cone only in generators, the other two points in this plane must be on a third generator, *i. e.*, $K_k \equiv K_3$. If we turn the plane of our G_4 on $P_1 P_3$ or $P_1 P_4$ instead of $P_1 O$, $C_{7,3}$ will of course be the same as before, but the trisecants will not pass through O'; they will determine the part of S_{15} aside from K_3, *i. e.*, an S_{12}. $C_{7,3}$ must therefore be common to K_3 and S_{12}. The double torsal generators will appear on both

surfaces and must be identical; this means that K_3 and S_{12} are tangent along these generators. $C_{7,8}$ is always a five-fold curve on S_{15}, and in this case it is counted once on K_3 and four times on S_{12}. The complete intersection, then, of K_3 and S_{12} is C_7 as a four-fold curve and the four double torsal generators each counted twice; *i. e.*,

$$4 \cdot 7 + 2 \cdot 4 = 36.$$

The genus of S_{12} is not greater than 7 nor less than 4. For, the plane section is of order 12, $D_{6,3}$ gives six double points, $C_{7,8}$ is four-fold on S_{12} and gives $7 \cdot \frac{4 \cdot 3}{2} = 42$ double points. $p = 1/2 \cdot 11 \cdot 10 - 6 - 42 = 7$. The C_4 factor of Δ_5 may have as many as three double points, which would give rise to an equal number of double torsal generators of S_{12}, and the minimum genus is therefore 4. Nothing of this sort can happen to K_3, which is, therefore, of genus 1.

[3] In order that $\Delta_5 \equiv C_3 \cdot C_2$, the F_i must be reducible to the form

$$f_i(x_1, x_2) + \psi_i(x_3, x_4, x_5) = 0.$$

By reason of the six double points of Δ_5, $D_{10} \equiv D_4 + 6\,D_1$. The six lines correspond to the six double points and there must be two factors of D_4 corresponding to the C_3 and C_2 of Δ_5. The genera must be 1 and 0; hence, $D_4 \equiv D_3 \cdot D_1$. This D_1 does not correspond to a double point of Δ_5 and is therefore not the vertex of a cone. Now project $D_{10,4}$ and $C_{8,4}$ from P_1 of the $C_{8,4}$ and build the surface of trisecants. $D_{3,3}$ and $D_{1,3}$ can not lie on the same ruled surface, for then the planes through P_1 defining the G_4''s and $C_{8,4}$ must cut $D_{10,4}$ twice, which is in general not possible. Therefore S_{15} is degenerate. $D_{3,4}$ is of genus 1 and is therefore a plane curve. This plane and the line D_1 are the fixed elements in the axial perspective which the above F_i shows must exist. Since from every point of D_1 four bisecants can be drawn to $C_{8,4}$, from every point of $C_{8,4}$ can be drawn a line cutting D_1 and $C_{8,4}$. Hence, when the projection is made from P_1, $D_{1,3}$ and $C_{7,8}$ will intersect on S_{15}. To find the order of that part of S_{15} on which $D_{1,3}$ lies, pass a plane through the line. It will cut $C_{7,8}$ in one point of $D_{1,3}$ and in six others. These six will be arranged in threes on the sides of a quadrilateral, for every generator must contain three points of $C_{7,8}$. The generators will also cut $D_{1,3}$. Thus each generator is cut by four others, which makes the order of the surface 6.* Therefore,

$$S_{15} \equiv S_6 \cdot S_9.$$

* Snyder: "On the Forms of Sextic Scrolls of Genus Greater than One," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXV (1903), pp. 261-268. See type II, p. 262.

C_7 can not be more than a double curve on S_6 nor more than three-fold on S_9. Then the double points of a plane section of S_6 are seven for C_7 and one for D_1, making eight. Therefore, $p=2$. The double points of a section of S_9 are twenty-one for C_7 and three for D_3. Therefore, $p=4$. If C_3 has a double point, S_9 will have one double torsal generator and its genus will be 3. As in the preceding case, all the lines of D_{10} that are vertices of cones will go into lines of tangency of the parts of S_{15}. The complete intersection then of S_6 and S_9 consists of C_7 counted six times and the six lines counted twice.

[4] If the F_i are of the form

$$a_i x_1^2 + b_i x_2^2 + \psi_i (x_3, x_4, x_5) = 0,$$

Δ_5 becomes $C_8 . C_1 . C_1' = 0$ and $S_{15} \equiv S_9 . K_3 . K_3' = 0$. C_1 and C_1' each determine four collinear double points of Δ_5, but the intersection of the two lines is thus counted twice and we have but seven double points. Corresponding to these there must be two tetrads of concurrent lines, the line of centers belonging to both. Let the lines be $a_1 a_2 \ldots . a_7$ and the centers O_1 and O_2. Call the line of centers a_4. Project into R_3 from a point of the normal curve. O_1' and O_2' will be the vertices of two cones K_3 and K_3' tangent along a_4'. The remainder of S_{15} must be S_9 tangent to K_3 along $a_1' a_2' a_3' a_4'$ and to K_3' along $a_4' a_5' a_6' a_7'$. Since K_3 and K_3' can have no nodal curves, the five-fold C_7 is a simple curve on each and three-fold on S_9. We see, then, that the complete intersection of S_9 and K_3 is C_7 counted three times and $a_1' a_2' a_3'$. For S_9 and K_3', C_7 and $a_5' a_6' a_7'$. K_3 and K_3' have C_7 and a_4' in common.

[5] If F_i are of the form

$$a_i x_1^2 + f_i (x_2, x_3) + \psi_i (x_4, x_5) = 0,$$

Δ_5 becomes $C_2 . C_2' . C_1$ and $S_{15} \equiv S_6 . S_6' . K_3 = 0$. C_1 determines four collinear double points of Δ_5. The corresponding lines of D_{10} will be $a_1 a_2 a_3 a_4$ concurrent at O. C_2 and C_2' determine four more double points of Δ_5 corresponding to which there will be four skew lines $a_5 a_6 a_7 a_8$ of D_{10}. The remainder of D_{10} is a C_2 consisting of the axes of the two axial perspectives under which F_i is invariant. Project as before into R_3 and the image of O will be the vertex of the cone S_3. C_7 will be double on S_6 and S_6' and simple on K_3. The intersection of S_6 and S_6' consists of C_7 counted four times and the double torsal generators, $a_5' a_6' a_7' a_8'$, which are lines of tangency. S_6 and K_3 intersect in C_7 counted twice and the two lines of tangency $a_1' a_2'$. S_6' and S_3 have C_7 and $a_3' a_4'$ in common.

[6] If F_i are of the form

$$a_i x_1^2 + b_i x_2^2 + c_i x_3^2 + f_i(x_4, x_5) = 0,$$

Δ_5 becomes $C_2 . C_1 . C_1' . C_1'' = 0$ and $S_{15} \equiv S_6 . K_3 . K_3' . K_3'' = 0$. C_1, C_1', C_1'' each determine four collinear double points of Δ_5. We will therefore have three tetrads of lines in D_{10}. Let the centers be O_1, O_2, O_3. Let a_1, a_2, a_3 correspond to the mutual intersections of the three lines and $a_4, a_5, \ldots, a_9$ correspond to the six points on C_2. Then, for the three tetrads and their center, we may use the notation

$$O_1(a_1, a_2, a_4, a_5), \quad O_2(a_1, a_3, a_6, a_7), \quad O_3(a_2, a_3, a_8, a_9),$$

where

$$a_1 \equiv O_1 O_2, \quad a_2 \equiv O_1 O_3, \quad a_3 \equiv O_2 O_3.$$

The remaining line of D_{10} does not correspond to a double point of Δ_5; hence, it is not the vertex of an R_4 cone. Projecting as before into R_3, O_1', O_2', O_3' are the vertices of K_3, K_3', K_3'' respectively. C_7 will be a double curve on S_6 and a simple curve on the cones.

K_3 and K_3' intersect in C_7 counted once and a_1'.
K_3 and K_3'' intersect in C_7 counted once and a_2'.
K_3' and K_3'' intersect in C_7 counted once and a_3'.
S_6 and K_3 intersect in C_7 counted twice and a_4', a_5'.
S_6 and K_3' intersect in C_7 counted twice and a_6', a_7'.
S_6 and K_3'' intersect in C_7 counted twice and a_8', a_9'.

[7] When Δ_5 consists of five straight lines, the ten intersections require that D_{10} shall consist wholly of straight lines. In this case F_i is reducible to

$$a_i x_1^2 + b_i x_2^2 + c_i x_3^2 + d_i x_4^2 + e_i x_5^2 = 0,$$

in which $\Sigma \lambda_i a_i$, $\Sigma \lambda_i b_i$, $\ldots$, $\Sigma \lambda_i e_i$, $(i = 1, 2, 3)$, are the five factors of Δ_5. As already shown, the ten lines of D_{10} constitute the simplex of reference in R_4 and intersect by fours in five points. When this configuration is projected from a point of $C_{8,4}$ into R_3, the five vertices of the simplex go into the vertices of the five K_3's into which S_{15} degenerates. The ten lines go into ten lines, joining these five vertices in pairs. Thus each cone passes through the vertices of the other four and any two of the five are tangent along their common generator. C_7 in each case completes the intersection which is of order 9.

8. Since the normal curve $C_{8,4}$ and the $C_{6,2}$ are in (1, 1) correspondence, the transformations that leave $C_{6,2}$ invariant may be obtained by finding those under which $C_{8,4}$ is invariant and then getting the corresponding transformation in R_2. The method in brief is this: Find a transformation T that leaves five points of R_2 invariant. Write the equation of the most general C_3 through them. It is the equation of the complete net of adjoining curves of C_6 and has five terms. Neglecting the constant coefficients, regard these five expressions in x, y, z as $\phi_1, \phi_2, \dots, \phi_5$, the homogeneous point coordinates in linear space of four dimensions. Find the three quadratic relations among them, two with numerical and the third with arbitrary coefficients. Apply T to the x, y, z coordinates and then find the corresponding transformation T_1 in ϕ_i. Next, determine the constants in the third quadratic relation so that it shall be invariant under T_1. This will be the equation of the C_6 that is invariant under T.

It must be kept in mind that Δ_5 will have transformations of the same order as those of C_6 if it does not remain fixed, point by point. C is defined by

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0,$$

and it is evident that the transformations that leave $C_{8,4}$ invariant must leave this net invariant. This can be done by collineations in ϕ which put F_i into F_i, leaving all points of $C_{8,4}$ fixed or permuted. Or F_i may go into linear combinations of F_i, which requires that Δ_5 have a corresponding linear transformation. In any case, Δ_5 as a whole must remain fixed because it is the condition for the double points of the system F which is invariant. Δ_5 is a quintic that can be written as a symmetric determinant. Such quintics have only the groups whose generators are G_2, G_3, G_4, G_5.* These we will now examine.

The cyclic groups G_7 and G_{13} need not be considered, as the associated curves can not be expressed in the above form.

A Linear G_2. Let $C_{6,2}$ have five distinct double points (0, 1, 0), (1, 1, 1), (1, 1, -1), (a, 0, c), (a, 0, $-c$). Write the equation of the general curve through these points. Then the five linearly independent adjoints by which we pass to R_4 may be written in the form

$$\begin{aligned}
&\phi_1 = c^2 x^3 + (a^2 - c^2)\, y z^2 - a^2 x z^2, \\
&\phi_2 = c^2 (x^2 y - y z^2), \quad \phi_4 = c^2 x^2 z + (a^2 - c^2)\, x y z - a^2 z^3, \\
&\phi_3 = c^2 (x y^2 - y z^2), \quad \phi_5 = c^2 (y^2 z - x y z).
\end{aligned}$$

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* Snyder: "Plane Quintic Curves Which Possess a Group of Linear Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), pp. 1–9.

Corresponding to $T_1 \equiv \begin{pmatrix} x\,y & z \\ x\,y & -z \end{pmatrix}$, we have

$$T_2 \equiv \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \phi_2 & \phi_3 & -\phi_4 & -\phi_5 \end{pmatrix}.$$

To find the quadratic identities, write the quadratic equation in the five ϕ_i and substitute the values of ϕ in terms of x, y, z. To make the equation an identity some of the coefficients will be zero, leaving

$$A\,\phi_1\phi_3 + B\,\phi_1\phi_5 + D\,\phi_2^2 + E\,\phi_2\phi_4 + F\phi_2\phi_5 + G\,\phi_3\phi_4 + K\phi_4\phi_5 = 0.$$

This equation is identically satisfied only in case $A = -D = -K$ and $B = E = \frac{c^2}{a^2 - c^2} F = -G$.

The result may be written in the form

$$A\,(\phi_1\phi_3 - \phi_2^2 - \phi_4\phi_5) + B\,(\phi_1\phi_5 + \phi_2\phi_4 + \frac{a^2 - c^2}{c^2}\phi_2\phi_5 - \phi_3\phi_4) = 0.$$

Hence, we have

$$F_1 \equiv \phi_1\phi_3 - \phi_2^2 - \phi_4\phi_5 = 0,$$
$$F_2 \equiv c^2\phi_1\phi_5 + c^2\phi_2\phi_4 + (a^2 - c^2)\,\phi_2\phi_5 - c\,\phi_3\phi_4 = 0.$$

F_3 may now be obtained by writing the general quadratic in ϕ_i, omitting two terms by means of F_1 and F_2, and imposing the condition that it shall be invariant under T_2. This means that F_3 must either be linear in ϕ_4, ϕ_5 or contain no such linear terms at all. The former will make z a factor of C_6 and is therefore inadmissible. Hence, ϕ_4 and ϕ_5 can appear only when they form quadratic terms.

$$F_3 \equiv A\,\phi_1^2 + B\,\phi_1\phi_2 + C\phi_2^2 + D\,\phi_2\phi_3 + E\,\phi_3^2 + F\phi_4^2 + G\,\phi_4\phi_5 + H\phi_5^2 = 0.$$

By T_2 the net

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$$

becomes

$$\lambda_1 F_1 + \lambda_2(-F_2) + \lambda_3 F_3 = 0;$$

hence, Δ_5 must have the transformation

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & -\lambda_2 & \lambda_3 \end{pmatrix}.$$

The forty-two inflections and one hundred twenty-four bitangents of $C_{6,2}$ are symmetrically placed with respect to $z = 0$.

A Linear G_3. Let the three vertices of the triangle of reference, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, be the double points of $C_{6,2}$. The general equation of a ϕ-curve through these points may be written

$$\begin{aligned} K&[(\omega-\omega^2)(x^2y-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)] \\ &+L[(\omega-\omega^2)(x^2z-\omega^2xyz)-(\omega-\omega^2)(yz^2-\omega^2xyz)] \\ &+M[(\omega-\omega^2)(xy^2-\omega^2xyz)-(\omega-\omega^2)(yz^2-\omega^2xyz)] \\ &+N[(\omega-\omega^2)(xz^2-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)] \\ &+P[(\omega-\omega^2)(y^2z-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)]=0. \end{aligned}$$

Transform to R_4 by

$$\begin{aligned} \phi_1 &= x^2y+yz^2+xyz, & \phi_4 &= xz^2+yz^2+xyz, \\ \phi_2 &= x^2z-yz^2, & \phi_5 &= y^2z+yz^2+xyz. \\ \phi_3 &= xy^2-yz^2, \end{aligned}$$

Corresponding to $T_1 \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$, we have

$$\left.\begin{aligned} \phi_1 &= \phi_2'+\phi_5', & \phi_3 &= -\phi_2, & \phi_5 &= \phi_2'+\phi_4'. \\ \phi_2 &= \phi_3'-\phi_2', & \phi_4 &= \phi_1'+\phi_2', \end{aligned}\right\} \equiv T_2.$$

Find the quadratic identities as before.

$$F_1 \equiv \phi_1\phi_4-\phi_2\phi_3-\phi_2\phi_5-\phi_4\phi_5=0.$$

By T_2 this becomes

$$F_2 \equiv \phi_1\phi_5+\phi_2\phi_5-\phi_3\phi_4-\phi_1\phi_4=0,$$

and F_2 becomes

$$F_2' \equiv \phi_4\phi_5+\phi_3\phi_4+\phi_2\phi_3-\phi_1\phi_5=-(F_1+F_2).$$

F_2' becomes F_1 by the same transformation; the cycle of order 3 is thus completed. F_3 may be found as before. The result is

$$\begin{aligned} F_3 \equiv A\phi_1^2&+B\phi_1\phi_2+C\phi_1\phi_3+D\phi_2^2+(A-D)\phi_2\phi_3+C\phi_2\phi_4 \\ &+(2A-B-C)\phi_2\phi_5+D\phi_3^2+(2A-B-C)\phi_3\phi_4+B\phi_3\phi_5 \\ &+A(\phi_4^2+\phi_5^2)=0. \end{aligned}$$

In this case, to make the net $\lambda_1F_1+\lambda_2F_2+\lambda_3F_3=0$ invariant, Δ_5 must have the group

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_2 & \lambda_1-\lambda_2 & \lambda_3 \end{pmatrix}.$$

The $C_{6,2}$ has 42 inflections and 124 bitangents. The whole number of inflections can be permuted in threes, but at least one bitangent must remain

fixed. The nodal tangents at $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ must remain fixed, because these points are invariant. To find what lines of the pencil $(1, \omega, \omega^2)$ are invariant, write the equation of the line through this point and (x, z, y). Then impose the condition on the coefficients that the line shall be invariant under T_1. The equation is

$$(\omega z_1 - \omega^2 y_1)x + (\omega^2 x_1 - z_1)y + (y_1 - \omega x_1)z = 0.$$

By T_1,

$$(\omega^2 x_1 - z_1)x + (y_1 - \omega x_1)y + (\omega z_1 - \omega^2 y)z = 0.$$

From these we find $(x_1 y_1 z_1) \equiv (1, 1, 1)$ or $(1, \omega^2, \omega)$. This means that one of the tangents at $(1, \omega, \omega^2)$ goes through $(1, 1, 1)$ and the other through $(1, \omega^2, \omega)$. By the same method we find that one of the tangents at $(1, \omega^2, \omega)$ goes through $(1, 1, 1)$ and the other through $(1, \omega, \omega^2)$; *i. e.*, the line joining these two double points is a bitangent. This leaves 123 bitangents to be permuted by the operations of G_3.

A linear G_4. Use $(1, 0, 0)$, $(1, 1, 1)$, $(1, i, -1)$, $(1, -1, 1)$, $(1, -i, -1)$ as double points. They are invariant under

$$\begin{pmatrix} x & y & z \\ x & iy & -z \end{pmatrix} \equiv T_1.$$

The equation of the general ϕ-curve through the above points is

$$B(x^2 y - y z^2) + C(x y^2 - z^3) + D(y^3 - x y z) + E(x^2 z - z^3) + F(x z^2 - y^2 z) = 0.$$

Now pass to R_4, make T_1 on ϕ_i and find the relation between ϕ_i and ϕ_i'.

$$\left.\begin{aligned} \phi_1 &= x^2 y - y z^2 = i(x^{2\prime} y' - y' z^{2\prime}) &&= i\phi_1' \\ \phi_2 &= x y^2 - z^3 = -(z^{2\prime} y' - z^{3\prime}) &&= -\phi_2' \\ \phi_3 &= y^3 - x y z = -i(y^{3\prime} - x' y' z') &&= -i\phi_3' \\ \phi_4 &= x^2 z - z^3 = -(x^{2\prime} z' - z^{3\prime}) &&= -\phi_4' \\ \phi_5 &= x z^2 - y^2 z = (x' z^{2\prime} - y^{2\prime} z') &&= \phi_5' \end{aligned}\right\} \equiv T_2.$$

By the usual method we find two identities,

$$F_1 \equiv \phi_1 \phi_5 + \phi_3 \phi_4 = 0,$$
$$F_2 \equiv \phi_1 \phi_3 - \phi_2^2 + \phi_2 \phi_4 + \phi_5^2 = 0.$$

By transforming the general equation by T_2, we find that the terms ϕ_1^2, $\phi_2 \phi_5$, ϕ_3^2, $\phi_4 \phi_5$ merely change sign. Hence, we may write

$$F_3 \equiv A\phi_1^2 + G\phi_2\phi_5 + H\phi_3^2 + N\phi_4\phi_5 = 0.$$

By T_2 the net

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$$

becomes

$$\lambda_1 i F_1 + \lambda_2 F_2 - \lambda_3 F_3 = 0.$$

Δ_5 must therefore be invariant under

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ -i\lambda_1 & \lambda_2 & -\lambda_3 \end{pmatrix}.$$

Now write F_3 in the form

$$A(x^2 y - y z^2)^2 + G(x y^2 - z^3)(x z^2 - y^2 z) + H(y^3 - x y z)^2 \\ + N(x^2 z - z^3)(x z^2 - y^2 z) = 0.$$

This is the equation of the $C_{6,2}$. At $y = 0$,

$$x z^3 [N x^2 - (G - N) z^2] = 0.$$

There are therefore three points of intersection at the vertex $(1, 0, 0)$. Since the equation contains only even powers of y this point is a cusp and $y = 0$ is the tangent. The presence of this cusp reduces the number of inflections to forty. These and the 124 bitangents are permuted in fours. $x = 0$ is an invariant tangent and $Nx^2 - (G - N)z^2 = 0$ defines two tangents through $(0, 1, 0)$ whose points of contact are on $y = 0$. The points on $y = 0$ interchange in pairs by T_1.

A Linear G_5. Let $(1, 1, 1)$, $(1, \theta, \theta^2)$, $(1, \theta^2, \theta^4)$, $(1, \theta^3, \theta)$, $(1, \theta^4, \theta^3)$ be the double points wherein $\theta^5 = 1$. The general equation of a C_3 is

$$A x^3 + B x^2 y + C x y^2 + D y^3 + E y^2 z + F x z^2 + G x^2 z + H y z^2 \\ + K x y z + L z^3 = 0.$$

If $(1, \theta, \theta^2)$ lies upon it,

$$(A + H) + (B + L)\theta + (C + G)\theta^2 + (D + K)\theta^3 + (E + F)\theta^4 = 0.$$

The remaining points give equations with the same coefficients in $(A + H)$, $(B + L)$, etc. Therefore since

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \theta & \theta^2 & \theta^3 & \theta^4 \\ 1 & \theta^2 & \theta^4 & \theta & \theta^3 \\ 1 & \theta^3 & \theta & \theta^4 & \theta^2 \\ 1 & \theta^4 & \theta^3 & \theta^2 & \theta \end{vmatrix} \neq 0,$$

$$A + H = B + L = C + G = D + K = E + F = 0.$$

The equation of the ϕ-curve then becomes

$$A(x^3 - yz^2) + B(z^3 - x^2y) + C(xy^2 - x^2z) + D(y^3 - xyz) + E(y^2z - xz^2) = 0.$$

Now pass to R_4 by the method used in the preceding cases. Corresponding to

$$T_1 \equiv \begin{pmatrix} x & y & z \\ x & \theta y & \theta^2 z \end{pmatrix},$$

we find

$$T_2 \equiv \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \theta\phi_2 & \theta^2\phi_3 & \theta^3\phi_4 & \theta^4\phi_5 \end{pmatrix}.$$

The terms ϕ_1^2, $\phi_1\phi_2$, $\phi_1\phi_3$, ϕ_2^2, $\phi_3\phi_4$, $\phi_2\phi_5$, $\phi_3\phi_5$, ϕ_4^2 and $\phi_4\phi_5$ contain terms in x, y, z that are not duplicated in quadratic combinations of ϕ_i. Hence, these can not appear in the identities. We have then to consider only six terms.

$$A\phi_1\phi_4 + B\phi_1\phi_5 + C\phi_2\phi_3 + D\phi_2\phi_4 + E\phi_3^2 + F\phi_5^2 = 0.$$

By substituting the values of ϕ_i in this equation, we find

$$A = C = F \quad \text{and} \quad B = D = E.$$

Hence,

$$F_1 \equiv \phi_1\phi_4 + \phi_2\phi_3 + \phi_5^2 = 0,$$
$$F_2 \equiv \phi_1\phi_5 + \phi_2\phi_4 + \phi_3^2 = 0.$$

By means of F_1 and F_2 replace two terms $\phi_1\phi_4$ and $\phi_1\phi_5$ by their equivalents in the general quadratic equation in ϕ_i. Make the transformation T_2. Only three terms remain invariant. Hence,

$$F_3 \equiv A\phi_1^2 + B\phi_2\phi_5 + C\phi_3\phi_4 = 0.$$

By T_2 the net $\Sigma\lambda_i F_i = 0$ becomes

$$\lambda_1\theta^3 F_1 + \lambda_2\theta^4 F_2 + \lambda_3 F_3 = 0.$$

This means that Δ_5 has the group

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \theta^2\lambda_1 & \theta\lambda_2 & \lambda_3 \end{pmatrix}.$$

In the above $C_{6,2}$ we find that $x = 0$ is a bitangent with ordinary contact at $(0, 0, 1)$ and four-point contact at $(0, 1, 0)$. Thus four invariant bitangents and two invariant inflections are accounted for. This leaves 120 bitangents and 40 inflections to be permuted in fives.

The points $(1, 1, 1)$, $(1, \theta, \theta^4)$, $(1, \theta^2, \theta^3)$, $(1, \theta^3, \theta^2)$, $(1, \theta^4, \theta)$ may be used as the basis of a G_5. They are invariant under a G_2 as well. Find the adjoint curves as before.

$$\begin{aligned} \phi_1 &= x^3 - xyz, & \phi_4 &= y^3 - xz^2, \\ \phi_2 &= x^2y - y^2z, & \phi_5 &= x^2z - yz^2. \\ \phi_3 &= z^3 - xy^2, \end{aligned}$$

Two identities are

$$F_1 \equiv \phi_1\phi_4 + \phi_2\phi_3 + \phi_5^2 = 0, \qquad F_2 \equiv \phi_1\phi_3 + \phi_4\phi_5 + \phi_2^2 = 0.$$

Then the most general quadratic form that is invariant under T_2 of the preceding G_5 is

$$F_3 \equiv A\phi_1^2 + B\phi_3\phi_4 + C\phi_2\phi_5 = 0.$$

The transformation in x, y, z is

$$T_3 \equiv \begin{pmatrix} x & y & z \\ x & \theta y & \theta^4 z \end{pmatrix}.$$

The result is

$$\theta^2\lambda_1\theta^3 F_1 + \theta^3\lambda_2 . \theta^2 F_2 + \lambda_3 F_3 = 0,$$

and the net has the same equation as before.

The curve $F_3 \equiv C_6$ has $x = 0$ for inflectional tangent at the vertices $(0, 0, 1)$ and $(0, 1, 0)$. These two inflections are invariant. $x = 0$ counts for four bitangents which are invariant. The remaining bitangents and inflections are permuted in fives.

We will now show the G_2 of this C_6. By $\begin{pmatrix} x\,y\,z \\ x\,z\,y \end{pmatrix}$, ϕ_2 and ϕ_5 interchange; also ϕ_3 and ϕ_4. This interchanges F_1 and F_2 but leaves F_3 invariant. If we transform Δ_6 by $\begin{pmatrix} \lambda_1\,\lambda_2\,\lambda_3 \\ \lambda_2\,\lambda_1\,\lambda_3 \end{pmatrix}$, the net will be invariant.

$$\lambda_1 F_2 + \lambda_2 F_1 + \lambda_3 F_3 = 0\,; \quad i.\,e., \quad \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0.$$

A necessary condition that Δ_6 be point by point invariant when F has a G_2 is that the G_2 be reducible to a change of sign, but it is not sufficient, for the transformation

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \phi_5 & \phi_4 & \phi_3 & \phi_2 \end{pmatrix},$$

mentioned above, may be put in the form

$$T \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & -x_4 & -x_5 \end{pmatrix}$$

by making the following linear transformation on ϕ_i:

$$\phi_1 = x_1, \quad \phi_2 + \phi_5 = x_2, \quad \phi_3 + \phi_4 = x_3, \quad \phi_4 - \phi_3 = x_4, \quad \phi_5 - \phi_2 = x_5.$$

Make this transformation on F_i.

$$F_1 \equiv 2x_1(x_3 + x_4) + (x_2 - x_5)(x_3 - x_4) + (x_2 + x_5)^2 = 0,$$
$$F_2 \equiv 2x_1(x_3 - x_4) + (x_2 + x_5)(x_3 + x_4) + (x_2 - x_5)^2 = 0,$$
$$F_3 \equiv A_1 x_1^2 + B_1(x_3^2 - x_4^2) + C_1(x_2^2 - x_5^2) = 0.$$

It is now evident that T interchanges F_1 and F_2. F_3 being unchanged, the G^2 of Δ_5 is the same as before. A G_{10} is the product of the above G_2 and G_5.

The following cases and curves were given by Wiman (W). When Δ_5 and $C_{8,4}$ have a G_μ, then, according to the seven cases mentioned in 6, we have groups whose orders are μ, 2μ, 2μ, 4μ, 4μ, 8μ, 16μ.

Case (a) The G_μ exists.
(b) A $G_{2\mu}$: Change of sign of one variable.
(c) A $G_{2\mu}$: Simultaneous change of signs of two variables.
(d) A $G_{4\mu}$: Two independent changes of sign.
(e) A $G_{4\mu}$: Change of sign of two pairs of variables.
(f) A $G_{8\mu}$: Three independent changes of sign.
(g) A $G_{16\mu}$: Four independent changes of sign.

The $G_{16\mu}$ exists when the F_i contain only the squares of the variables. In illustration of (f), we have a G_{192} composed of G_8 consisting of changes of sign and G_{24} the octahedral group shown by the factors $x_4 x_5 (x_4^2 + x_5^2)(x_4^2 - x_5^2)$. The curve is defined by

$$F_1 \equiv x_1^2 + x_4^2 + x_5^2 = 0,$$
$$F_2 \equiv x_2^2 + x_4^2 - x_5^2 = 0,$$
$$F_3 \equiv x_3^2 + x_4 x_5 = 0.$$

The curve with a G_{64} is defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$
$$x_1^2 + i x_2^2 - x_3^2 - i x_4^2 = 0,$$
$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0, \quad (i^2 = -1).$$

The G_{64} is composed of G_{16} and the cyclic G_4 on the variables $x_1 x_2 x_3 x_4$.

For G_{96} we have

$$x_1^2 + x_4^2 + x_5^2 = 0,$$
$$x_2^2 + j x_4^2 + j^2 x_5^2 = 0,$$
$$x_3^2 + j^2 x_4^2 + j x_5^2 = 0, \quad (j^3 = 1).$$

The components of this group are G_{16},

$$G_2 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_2 & x_5 & x_4 \end{pmatrix}, \qquad G_3 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_1 & x_2 & j x_4 & j^2 x_5 \end{pmatrix}.$$

For G_{160} we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$
$$x_1^2 + e x_2^2 + e^2 x_3^2 + e^3 x_4^2 + e^4 x_5^2 = 0,$$
$$e^4 x_1^2 + e^3 x_2^2 + e^2 x_3^2 + e x_4^2 + x_5^2 = 0, \quad (e^5 = 1).$$

The components of the group are G_{16},

$$G_2 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix}, \qquad G_5 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_5 & x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

9. (b) If a $C_{6,2}^{(5)}$ has a linear g_3^1, it must have a triple point.* By means of quadric inversion having the P_3 and the two P_2 for fundamental points, the curve is reduced to a nodal quintic. The adjoint curves are conics through the node (0, 0, 1). The general equation of these conics is

$$A x^2 + B x y + C y^2 + D x z + E y z = 0. \tag{26}$$

Pass to R_4 by the transformation

$$\phi_1 = x^2, \quad \phi_2 = x y, \quad \phi_3 = y^2, \quad \phi_4 = x z, \quad \phi_5 = y z. \tag{27}$$

Then three quadratic forms exist which are identities in x, y, z.

$$\phi_1 \phi_3 - \phi_2^2 = 0, \quad \phi_1 \phi_5 - \phi_2 \phi_4 = 0, \quad \phi_2 \phi_5 - \phi_3 \phi_4 = 0. \tag{28}$$

Three general quadrics in R_4 determine a $C_{8,4}$, but the set in (28) have a ruled hypersurface S in common for

$$\phi_1/\phi_2 = \phi_2/\phi_3 = \phi_4/\phi_5 = x/y = \lambda, \tag{29}$$

and $\phi_1 = \lambda \phi_2$, $\phi_2 = \lambda \phi_3$, $\phi_4 = \lambda \phi_5$ define a straight line in R_4 which is the image of the line $x = \lambda y$ in R_2. Therefore, S is the R_4 image of the plane generated

* Snyder: "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), pp. 10–18. See Art. 3.

by the pencil of lines through (0, 0, 1). The point $x=y=0$ has for its image the line $\phi_1=\phi_2=\phi_3=0$. All the lines, $x=\lambda y$, pass through the node. Hence their images, the generators of S, must cut the image of the node; *i. e.*, $\phi_1=\phi_2=\phi_3$ is the directrix. Each line of the nodal pencil cuts C_5 in three variable points. Therefore, each generator is a trisecant of the R_4 image of C_5 which lies on S. Since the node is a double point of C_5, its image must have two points on the directrix; *i. e.*, the directrix is a bisecant. Since the whole plane is uniquely pictured on S, $C_{5,2}$ and $C_{8,4}$ are invariant under the same groups of transformations. When C_5 is invariant, the system of adjoint conics must go into itself. But any of these adjoints can be expressed linearly in terms of the five and in no other way; *i. e.*, the R_4 transformations will all be linear. By these, straight lines will go into straight lines. This means that the generators of S will simply be permuted among themselves. Thus the lines of the nodal pencil will also be permuted. Moreover, adjoint conics must go into adjoint conics. When one factor of a conic is a line through the node, the other factor is some straight line which must go into a straight line by the transformation. This is possible only by collineations. Therefore, the only birational transformations under which a nodal quintic curve remains invariant are linear. Further, we may take $x=0$, $y=0$ as the nodal tangents. If C_5 is invariant, these tangents must either remain fixed or interchange. Hence, the transformations will be of the form

$$\begin{pmatrix} x & y & z \\ \lambda x & \mu y & \nu z \end{pmatrix} \text{ or } \begin{pmatrix} x & y & z \\ \lambda y & \mu x & \nu z \end{pmatrix},$$

where λ, μ, ν are roots of unity.

A G_2. A nodal C_5 will go into itself by a harmonic homology if it is symmetric about a line through the node. This line $x=0$ can not be a nodal tangent, for such a condition would call for a third tangent as the image of the second one. It would also require that $x=0$ be a tacnodal tangent. We can write the equation of the required C_5 with $x-y=0$ and $x+y=0$ as nodal tangents by omitting the odd powers of x from the general equation. Let the curve go through the vertices of the triangle; then we have

$$z^3(x^2-y^2)+z^2(Ax^2y+By^3)+z(Cx^4+Dx^2y^2+Ey^4)+Fx^4y+Gx^2y^3=0. \quad (31)$$

The center O of this homology is at the vertex (1, 0, 0). The coefficient of x^4 in (31) is the tangent to C_5 at O, $Cz+Fy=0$. This line must cut C_5 in an

even number of points apart from O. Hence, it must have either three- or five-point contact at O. In either case an odd number of inflections will be invariant. Since by Plücker's formulas the total number is thirty-nine, the remaining ones can be interchanged in pairs. The line $Cz + Fy = 0$ will count for two bitangents or none according as it has five- or three-point contact. There are forty-two in all, and in either case they can be interchanged in pairs. The transformation is

$$\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}.$$

No perspective whose period is of odd order can exist with a nodal line as axis, for it would require an odd number of tangents at the node. None of order 4 can exist, for, retaining only those x-terms that contain x^4, the equation becomes

$$-z^3y^2 + Bz^2y^3 + Czx^4 + Ezy^4 + Fx^4y = 0.$$

At the vertex (0, 0, 1) this shows $x = 0$ to be a tacnodal tangent which reduces the genus of C_5.

A G_3. We may write the equation of a $C_{5,2}^{(5)}$ that is invariant under the operations of a linear G_3 by omitting the z and z^2 terms from the general equation. The equation can be put in the form

$$z^3(x^2 - \alpha^2 y^2) + (ax + by)(cx + dy)(ex + fy)(gx + hy)(mx + ny) = 0. \quad (32)$$

The factors of the second term in this equation show five inflectional tangents which are concurrent at (0, 0, 1) and whose points of contact are on $z = 0$. If the second factor in the first term vanishes, the equation becomes $x^5 = 0$ or $y^5 = 0$. Hence, the nodal tangents $x + \alpha y = 0$ and $x - \alpha y = 0$ must vanish to the fifth order at the node. Since each of them cuts the branch to which the other is tangent only once, their order of contact must be 3. The five inflectional tangents count for ten tangents through (0, 0, 1). The nodal tangents count for four each. But the class of the C_5 is 18; hence, all the tangents through the node are accounted for. These will remain fixed under the G_3 given below. The curve has thirty-nine inflections. We have just seen that five of them are on the line $z = 0$. The four-point contacts of the nodal tangents count for four inflections. These nine are invariant, leaving thirty to be permuted

in threes. Two of the ninety-two bitangents are accounted for by the nodal tangents. The remaining ones will be permuted. The transformation is

$$\begin{pmatrix} & y & z \\ & y & \omega z \end{pmatrix} \text{ or } \begin{pmatrix} & y & z \\ & y & \omega^2 z \end{pmatrix}.$$

If the constants are such that (32) is symmetric in x and y, we have

$$z^2(x^2+y^2) + A(x^4y + xy^4) + B(x^2y + xy^2) + C(x^5+y^5) = 0. \tag{33}$$

This curve admits of a G_2 as well as a G_3. Hence, it is invariant under the following G_6,

$$\begin{pmatrix} x & y & z \\ y & x & \omega z \end{pmatrix} \text{ or } \begin{pmatrix} x & y & z \\ y & x & \omega^2 z \end{pmatrix}.$$

Nine inflections and two bitangents will be invariant, as before. The others will be permuted in sixes.

A group G_5 is of the form

$$\begin{pmatrix} x & y & z \\ \theta x & \theta^4 y & z \end{pmatrix}.$$

The equation of the invariant C_5 is

$$z^3xy + azx^2y^2 + bx^5 + cy^5 = 0. \tag{34}$$

This curve has the nodal tangents $x=0$ and $y=0$, each having four-point contact. Thus these two lines count for two bitangents, eight simple tangents through $(0, 0, 1)$ and four inflectional tangents. The remaining ninety bitangents, ten simple tangents through the node and thirty-five inflectional tangents will be grouped in fives. If (l, m, n) is a point of inflection, four others are associated with it, namely $(\theta l, \theta^4 m, n)$, $(\theta^2 l, \theta^3 m, n)$, $(\theta^3 l, \theta^2 m, n)$, $(\theta^4 l, \theta m, n)$. These five points lie on the conic $n^2xy - lmz^2 = 0$. Therefore, the thirty-five points of inflection are arranged by fives on seven conics tangent to each other at their points of intersection with $z=0$. If $n=0$, the conic becomes $z^2=0$, which cuts each of the conics twice at their common points of tangency. The point $(1, -(b/c)^{1/5}, 0)$ is a point of inflection. The inflectional tangent is

$$x + \left(\frac{c}{b}\right)^{1/5} \cdot y + \frac{a}{5\,b^{3/5}\,c^{2/5}}\,z = 0. \tag{35}$$

Hence, $z=0$ cuts the curve in five points of inflection.

If in (34) $b=c$, the equation becomes

$$z^3xy + azx^2y^2 + b(x^5+y^5) = 0. \tag{36}$$

The C_5 is now invariant under

$$\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix},$$

and therefore has a G_2 in addition to the G_5 just given. We must now account for one invariant inflection out of the thirty-five. It will also be necessary to fix four more so that the number will be divisible by both five and two. Let $(c/b)^{1/5} = \theta$, where $\theta^5 = 1$. Then the point $(1, -(b/c)^{1/5}, 0)$ becomes $(\theta, -1, 0)$. (35) becomes

$$x + \theta y + \frac{a}{5b}\theta^3 z = 0. \tag{37}$$

This tangent and its point of contact are invariant under the transformation

$$\begin{pmatrix} x & y & z \\ \theta y & \theta^4 x & z \end{pmatrix},$$

obtained by combining G_2 and G_5. Though it is formally a G_{10}, the transformations of this group interchange the points in pairs. Hence, the other four inflections on z are interchanged. This leaves thirty inflections. The ten simple tangents through (0, 0, 1) and the ninety bitangents remain as in G_5. (36) is therefore invariant under the above G_{10}. Since (36) is only a special case of (34), we may apply G_5 to any three distinct points of inflection (a, b, c), (a', b', c'), (a'', b'', c'') and obtain twelve other inflections. The fifteen points will be distinct, provided no one of the first three goes into either of the others by the successive transformations. Next apply G_2 to each of these. We thus get fifteen others, which completes the thirty.

By G_5:			By G_2:		
a	b	c	b	a	c
θa	$\theta^4 b$	c	$\theta^4 b$	θa	c
$\theta^2 a$	$\theta^3 b$	c	$\theta^3 b$	$\theta^2 a$	c
$\theta^3 a$	$\theta^2 b$	c	$\theta^2 b$	$\theta^3 a$	c
$\theta^4 a$	θb	c	θb	$\theta^4 a$	c

These ten points lie on the conic $c^2 xy - abz^2 = 0$. Hence, they constitute the complete intersection of our C_5 with the above conic. A similar set of points may be found from (a', b', c') and (a'', b'', c''). If we denote the three conics by C_2, C_2' and C_2'', the thirty inflections are the complete intersections of C_5 with the family of sextics $C_2 . C_2' . C_2'' + K z C_5 = 0$.

We will now combine G_3 and G_5 for a G_{15}. To make the curve invariant under

$$\begin{pmatrix} x & y & z \\ \theta x & \theta^4 y & z \end{pmatrix},$$

let $a=0$ in (34). The equation is

$$z^3 x y + b x^5 + c y^5 = 0. \tag{38}$$

As in G_5, the nodal tangents count for two bitangents, eight simple tangents through the node O and four inflectional tangents. $x+(c/b)^{1/5} y=0$ is an inflectional tangent. It is independent of ω in the G_{15} and therefore has but four images. These five inflectional tangents account for the remaining ten tangents through O. There are then thirty inflections and ninety bitangents to be permuted by G_{15}. When $b=c$, we have a G_{30} defined by

$$\begin{pmatrix} x & y & z \\ \theta y & \theta^4 x & \omega z \end{pmatrix}.$$

The tangent $x+(c/b)^{1/5} y=0$ becomes $x+\theta y=0$. Its point of contact is $(\theta, -1, 0)$, which is invariant. By G_5 the four images are $(\theta^2, -\theta^4, 0)$, $(\theta^3, -\theta^3, 0)$, $(\theta^4, -\theta^2, 0)$, $(1, -\theta, 0)$. G_{30} interchanges these four in pairs as in G_{10}.

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